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The entrainment interface

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A theory is developed to describe the evolution of the entrainment interface in turbulent flow, in which the surface is convoluted by the large-scale eddies of the motion and at the same time advances relative to the fluid as a result of the micro-scale entrainment process. A pseudo-Lagrangian description of the process indicates that the interface is characterized by the appearance of 'billows' of negative curvature, over which surface area is, on average, being generated, separated by re-entrant wedges (lines of very large positive curvature) where surface area is consumed. An alternative Eulerian description allows calculation of the development of the interfacial configuration when the velocity field is prescribed. Several examples are considered in which the prescribed velocity field in the z direction is of the general form $w = Wf(\mathbf{x} - \mathbf{U}t)$, where the *maximum* value of the function f is unity. These indicate the importance of *leading points* on the surface which are such that small disturbances in the vicinity will move away from the point in all directions. The necessary and sufficient condition for the existence of one or more leading points on the surface is that $U \leqslant V$, the speed of advance of an element of the surface relative to the fluid element at the same point. The existence of leading points is accompanied by the appearance of line discontinuities in the surface slope re-entrant wedges. In these circumstances, the overall speed of advance of the convoluted surface is found to be $W + (V^2 - U^2)^{\frac{1}{2}}$, where W is the maximum outwards velocity in the region; this result is independent of the distribution f.

When the speed U with which an 'eddy' moves relative to the outside fluid is greater than the speed of advance V of an element of the front, the interface develops neither leading points nor discontinuities in slope; the amplitude of the surface convolutions and the overall entrainment speed are both reduced greatly. In a turbulent flow, therefore, the large-scale motions influencing entrainment are primarily those that move slowly relative to the outside fluid (with relative speed less than V). The experimental results of Kovasznay, Kibens & Blackwelder (1970) are reviewed in the light of these conclusions. It appears that in their experiments the entrainment speed V is of the order fifteen times the Kolmogorov velocity, the large constant of proportionality being apparently the result of augmentation by micro-convolutions of the interface associated with small and meso-scale eddies of the turbulence.

1. Introduction

For many years, it has been known that in free turbulent flows such as jets, wakes and boundary layers, there is a sharp but convoluted interface between the region where the flow is turbulent and the external region of irrotational motion. Early experiments by Corrsin (1943) and Townsend (1948) showed that the signals from a hot wire probe placed near the outer edge of such a flow are of two distinct types, one indicating slow and smoothly varying velocity fluctuations indispersed among sections where the velocity fluctuates much more rapidly as turbulent billows are swept past the probe. The fraction of the total time that the signal is 'turbulent' is called the intermittency factor γ; as a result of the large-scale random convolutions of the interface, γ varies smoothly from unity well inside the flow to zero well outside.

More extensive studies were published by Corrsin & Kistler (1955). They pointed out that in a fluid of constant density, vorticity can be acquired by a fluid element outside the turbulent interface in the first place only by viscous diffusion of vorticity, but that once such an element has some vorticity, this can be amplified by the straining induced by the neighbouring turbulence. In time, the root-mean-square vorticity of such fluid elements becomes comparable with that of the turbulence itself, and their incorporation into the turbulence is then complete. Corrsin & Kistler argued that the thickness of the 'interface' and the rate of advance of an element of area are both governed by the balance between the viscous diffusion of vorticity and the mean-square rate of straining induced by the turbulence. The former is determined by the molecular viscosity ν itself, while the latter is proportional to (ϵ_0/ν), where ϵ_0 is the rate of energy dissipation in the turbulence near the interface. The thickness of the 'interface' is then, on similarity grounds, necessarily proportional to the Kolmogorov microscale $(\nu^3/\epsilon_0)^{\frac{1}{4}}$ and the speed of advance u_a to the Kolmogorov velocity scale $u_\nu = (\epsilon_0 \nu)^{\frac{1}{4}}$.

However, it is well known that the mean position of the interface (characterized by the point $\gamma = 0{\cdot}5$, say) advances towards the non-turbulent region with a speed u_e that may be more characteristic of the velocities of the energy-containing eddies of the motion than is u_ν. Indeed, were this not so, self-preservation of mean properties of flows such as jets would not be possible. The mean rate of incorporation of non-turbulent fluid per unit projected area can be expressed either as u_e, or as u_a times the mean interfacial area per unit projected area. If the position of the interface is specified by the possibly multiple valued function $z = \zeta(x, y, t)$, then

$$u_e = u_a\overline{[1 + (\nabla\zeta)^2]^{\frac{1}{2}}}, \tag{1.1}$$

but it was not clear, to this writer at least, how the interfacial convolutions adjusted themselves to preserve this equality.

A rather different view has also been suggested from time to time. It is that volumes of fluid, initially outside the turbulent region, are sporadically engulfed by the turbulence to be digested at leisure by the same processes of viscous diffusion of vorticity and amplification by straining. If this were so, part of the entrainment interface would be internal to the turbulence and detached from the external front; the mean area per unit projected area of the external interface would be less than if the interface were continuous. It is difficult either to confirm or deny this view by observation: measurements in a plane normal to the mean interface that show apparently isolated patches of irrotational motion may simply represent a section through a tongue of fluid not yet turbulent but not detached from the main volume.

This view of entrainment is an extreme one and possibly rather implausible from a dynamical point of view. The analysis in later sections of this paper points strongly to an intermediate description in which the entrainment is augmented by micro-convolutions of the interface on scales of the same order as the interface thickness itself. This enhancement of the rate of advance of an element of area of the interface implies that the mean-square slope of the surface, when observed on a large scale, is much less than would be required by (1.1), in which u_a is identified with the Kolmogorov microscale.

Recently, the structure of the entrainment interface was the subject of two fine experimental studies, by Kovasznay, Kibens & Blackwelder (1970) and by Kaplan & Laufer (1968). The measurements included the mean velocity distributions in which averages were taken separately over intervals in which the fluid motion at a fixed point was either turbulent or non-turbulent, the space–time history of the turbulent bulges and the characteristic large-scale velocity distributions associated with the bulges. One important finding of Kovasznay *et al.* was that the velocity appears to be continuous across the entrainment interface. This seems to exclude yet another suggestion concerning the detailed mechanics of entrainment that near-discontinuities in tangential velocity might arise, leading to sporadic local instability and breakdown. No doubt something of the kind does take place in stratified fluids when buoyancy differences associated with a sharp change in density across the interface render it essentially flat: the experiments of Thorpe (1969) attest to this. In a homogeneous fluid, however, the absence of a vortex sheet or even of a particularly sharp velocity gradient in the experiments of Kovasznay *et al.* suggests that sporadic instability is not a significant contributor to turbulent entrainment.

If we accept the view that entrainment is the result of vorticity diffusion by viscosity, subsequent amplification by straining augmented by micro-scale convolutions of the interface, and all processes associated with the smallest scales of motion, then it follows that the detailed mechanics of entrainment are statistically independent of the large-scale motions that convolute the surface. This idea of the statistical independence of large and small-scale motions is at the heart of the theory of local similarity in turbulence; in considering turbulent entrainment, it enables us to specify separately the velocity of advance of the interface and the large-scale motions deforming it. Convenient and familiar though the idea may be, it is important to recognize that it may not be correct in this particular context. The small-scale motion near the edge of the entrainment interface is hardly a random sampling of the small-scale structure in the turbulence as a whole, and it would not be surprising to find that systematic variations in the small structure were associated with the large structure convoluting the surface. For example, small-scale eddies near the crest of a billow may be different from those in a trough. Nevertheless, at this stage, it seems wisest simply to parametrize the encroachment associated with the small-scale structure by a single velocity V, independent of the larger-scale motion, and to explore the consequences of this model before adding additional complexity.

It might be noted that the kinematics of a deforming, advancing turbulent entrainment interface are very similar to those of a turbulent flame front, which has been the subject of an independent study by I. Howells (private communication).

2. A pseudo-Lagrangian description

Although the entrainment interface has a thickness of the order of the Kolmogorov microscale $(\nu^3/\epsilon_0)^{\frac{1}{4}}$, this thickness is very small compared with the length scale of the energy-containing eddies convoluting the interface, provided the Reynolds number of the turbulent flow as a whole is sufficiently large. It is therefore convenient to represent the interface as a geometrical surface that is convoluted by the random velocity field induced by the large eddies of the turbulence, which at the same time advances normal to itself with speed $u_a = V$ into the region of irrotational motion.

Suppose $\mathbf{r}(\mathbf{x}, t)$ represents the position vector of a point on the interface and moving with it, the instantaneous location at time $t = 0$ being $\mathbf{r}(\mathbf{x}, 0) = \mathbf{x}$. The velocity of the entrainment interface is

$$\frac{d\mathbf{r}(\mathbf{x})}{dt} = \mathbf{u}(\mathbf{x}) + V\mathbf{n}(\mathbf{x}), \tag{2.1}$$

where $\mathbf{n}$ is the unit normal to the surface away from the turbulent region. A neighbouring point on the interface, situated at $\mathbf{x}+\mathbf{l}$, moves with velocity

$$\begin{aligned}\frac{d}{dt}\mathbf{r}(\mathbf{x}+\mathbf{l}) &= \mathbf{u}(\mathbf{x}+\mathbf{l}) + V\mathbf{n}(\mathbf{x}+\mathbf{l}),\\ &= \mathbf{u}(\mathbf{x}) + \mathbf{l}.\nabla\mathbf{u}(\mathbf{x}) + V\mathbf{n}(\mathbf{x}) + V\mathbf{l}.\nabla\mathbf{n}(\mathbf{x}) + \ldots,\end{aligned}$$

so that the rate of change of the line element $\mathbf{l}$ is

$$\begin{aligned}\frac{d\mathbf{l}}{dt} &= \frac{d}{dt}\mathbf{r}(\mathbf{x}+\mathbf{l}) - \frac{d}{dt}\mathbf{r}(\mathbf{x}),\\ &= \mathbf{l}.\nabla\mathbf{u} + V\mathbf{l}.\nabla\mathbf{n}.\end{aligned} \tag{2.2}$$

In the notation of Cartesian tensors,

$$\frac{dl_i}{dt} = l_j\frac{\partial u_i}{\partial x_j} + Vl_j\frac{\partial n_i}{\partial x_j}. \tag{2.3}$$

When (2.3) is contracted with l_i, it follows that

$$\begin{aligned}l_i\frac{dl_i}{dt} = \frac{d}{dt}\tfrac{1}{2}l^2 &= l\frac{dl}{dt}\\ &= l_il_j\frac{\partial u_i}{\partial x_j} + Vl_il_j\frac{\partial n_i}{\partial x_j},\end{aligned}$$

and, if $\nu_i = l_i/l$ is a unit vector in the direction of the line element in the interfacial surface, then

$$\frac{1}{l}\frac{dl}{dt} = \nu_i\nu_j\frac{\partial u_i}{\partial x_j} - LV, \tag{2.4}$$

where, by definition,
$$L = n_i \nu_j \frac{\partial \nu_i}{\partial x_j} = -\nu_i \nu_j \frac{\partial n_i}{\partial x_j}, \tag{2.5}$$
since $n_i \nu_i = 0$.

The quantity L is one of the fundamental magnitudes of second order in the differential geometry of surfaces (Weatherburn 1955). If $\boldsymbol{\nu}'$ is a unit vector in the surface normal to both $\boldsymbol{\nu}$ and $\mathbf{n}$ at time $t = 0$, directed so that $\mathbf{n} = \boldsymbol{\nu} \times \boldsymbol{\nu}'$, then a second such magnitude is
$$N = n_i \nu'_j \frac{\partial \nu'_i}{\partial x_j} = -\nu'_i \nu'_j \frac{\partial n_i}{\partial x_j}, \tag{2.6}$$
since also $n_i \nu'_i = 0$. Further, if dl, dl' are elements of length in the directions of $\boldsymbol{\nu}$, $\boldsymbol{\nu}'$, since $\nu_i = \partial r_i/\partial l$,
$$\nu'_j \frac{\partial \nu_i}{\partial x_j} = \frac{\partial \nu_i}{\partial l'} = \frac{\partial^2 r_i}{\partial l\, \partial l'} = \frac{\partial \nu'_i}{\partial l} = \nu_j \frac{\partial \nu'_i}{\partial x_j},$$
and it follows that the quantity
$$\begin{aligned} M &= n_i \nu'_j \frac{\partial \nu_i}{\partial x_j} = -\nu_i \nu'_j \frac{\partial n_i}{\partial x_j} \\ &= n_i \nu_j \frac{\partial \nu'_i}{\partial x_j} = -\nu'_i \nu_j \frac{\partial n_i}{\partial x_j}. \end{aligned} \tag{2.7}$$
As the surface is distorted, the unit vectors $\boldsymbol{\nu}$ and $\boldsymbol{\nu}'$ will not remain perpendicular, but at the initial instant we have the fundamental magnitudes of the first order,
$$E = \nu_i \nu_i = 1, \quad F = \nu_i \nu'_i = 0, \quad G = \nu'_i \nu'_i = 1, \quad H^2 = EG - F^2 = 1 \tag{2.8}$$
at $t = 0$, in Weatherburn's notation. Finally, two convenient relations follow from the fact that the spatial rate of change of the unit normal is perpendicular to itself, and so in the plane of $\boldsymbol{\nu}$ and $\boldsymbol{\nu}'$
$$\left.\begin{aligned} \nu'_j \frac{\partial n_i}{\partial x_j} &= -\nu_i M - \nu'_i N, \\ \nu_j \frac{\partial n_i}{\partial x_j} &= -\nu'_i M - \nu_i L. \end{aligned}\right\} \tag{2.9}$$

With these expressions from differential geometry, the time rates of change of the unit vectors can be found simply. Since $l_i = l\nu_i$,
$$l \frac{d\nu_i}{dt} = \frac{dl_i}{dt} - \nu_i \frac{dl}{dt},$$
and, from (2.3) and (2.4),
$$\begin{aligned} \frac{d\nu_i}{dt} &= \nu_k \frac{\partial u_i}{\partial x_k} + V l_k \frac{\partial n_i}{\partial x_k} - \nu_i \nu_j \nu_k \frac{\partial u_j}{\partial x_k} + \nu_i L V, \\ &= \nu_k (\delta_{ij} - \nu_i \nu_j) \frac{\partial u_j}{\partial x_k} - \nu'_i M V, \end{aligned} \tag{2.10}$$
from (2.9). Similarly,
$$\frac{d\nu'_i}{dt} = \nu'_k (\delta_{ij} - \nu'_i \nu'_j) \frac{\partial u_j}{\partial x_k} - \nu_i M V. \tag{2.11}$$

These relations can be expressed alternatively as

$$\left.\begin{aligned}\frac{d\nu_i}{dt} &= \left\{\nu_k n_j \frac{\partial u_j}{\partial x_k}\right\} n_i + \left\{\nu_k \nu'_j \frac{\partial u_j}{\partial x_k} - MV\right\} \nu'_i,\\ \frac{d\nu'_i}{dt} &= \left\{\nu'_k n_j \frac{\partial u_j}{\partial x_k}\right\} n_i + \left\{\nu'_k \nu_j \frac{\partial u_j}{\partial x_k} - MV\right\} \nu_i.\end{aligned}\right\} \qquad (2.12)$$

The time rate of change of the unit normal $\mathbf{n} = \boldsymbol{\nu} \times \boldsymbol{\nu}'$ to the surface is given by

$$\begin{aligned}\frac{dn_i}{dt} &= \epsilon_{ijk} \frac{d\nu_j}{dt} \nu'_k + \epsilon_{ijk} \nu_j \frac{d\nu'_k}{dt}\\ &= -n_l(\nu_i \nu_m + \nu'_i \nu'_m) \frac{\partial u_l}{\partial x_m},\end{aligned} \qquad (2.13)$$

after simple calculation.

The rate of change of area of an element of the moving interface can now be found readily. Since

$$\mathbf{n}A = \mathbf{l} \times \mathbf{l}', \quad A = \mathbf{n}.\mathbf{l} \times \mathbf{l}'$$

and

$$\frac{dA}{dt} = \frac{d\mathbf{n}}{dt}.\mathbf{l} \times \mathbf{l}' + \mathbf{n}.\frac{d\mathbf{l}}{dt} \times \mathbf{l}' + \mathbf{n}.\mathbf{l} \times \frac{d\mathbf{l}'}{dt},$$

and, from (2.3) and (2.13), it follows, after some re-arrangement, that

$$\frac{1}{A}\frac{dA}{dt} = (\nu_j \nu_l + \nu'_j \nu'_l) \frac{\partial u_j}{\partial x_l} - VJ, \qquad (2.14)$$

where, since $\mathbf{l}$ and $\mathbf{l}'$ are perpendicular at the instant concerned, the mean curvature

$$J = L + N. \qquad (2.15)$$

The first term on the right of (2.14) represents the net rate of surface divergence of the fluid elements that instantaneously lie in the entrainment interface, a quantity that can be represented by θ, so that

$$\frac{1}{A}\frac{dA}{dt} = \theta - VJ. \qquad (2.16)$$

In this form, the expression is independent of our choice of directions for the unit vectors $\boldsymbol{\nu}$ and $\boldsymbol{\nu}'$, and holds throughout the history of the element of interfacial surface.

When $J < 0$ (i.e. when a 'billow' of the turbulence extends into the irrotational fluid), the advance of the interface contributes to the generation of surface area. Moreover, in such regions the surface divergence of fluid elements would be expected to be positive, the fluid moving outwards generally from the top of the billow. Consequently, we are led to the general statement that outward billows of turbulence are, on average, sources of area of the entrainment interface. On the other hand, if J is sufficiently positive regardless of the local value of θ, $dA/dt < 0$ and surface area of the entrainment interface is destroyed. This corresponds geometrically to regions in 'valleys' between the turbulent hummocks; these regions tend to be sinks of surface area. It is clear, even at this stage, that the kinematics of the entrainment interface depend heavily on the local surface curvature, whose development will be considered in detail § 3.

Before we do this, however, it might be noted that the rate of change of the angle between the two line elements on the interface is specified by the rate of change of $F = \boldsymbol{\nu}.\boldsymbol{\nu}'$. At the initial instant, $F = 0$ since $\boldsymbol{\nu}$ and $\boldsymbol{\nu}'$ are perpendicular, but

$$\frac{dF}{dt} = \nu_i \frac{d\nu_i'}{dt} + \nu_i' \frac{d\nu_i}{dt}$$

$$= (\nu_i \nu_j' + \nu_i' \nu_j) \frac{\partial u_i}{\partial x_j} - 2MV, \tag{2.17}$$

from (2.12). The first term on the right of this expression represents twice the surface shear strain rate in the $\boldsymbol{\nu}, \boldsymbol{\nu}'$ directions, σ, say. The relative angular velocity ω of two perpendicular line elements in the entrainment interface is therefore

$$\omega = -2\sigma + 2MV. \tag{2.18}$$

3. Curvature of the entrainment interface

The first (or mean) curvature of the surface, the sum of the two principal curvatures, is given generally by

$$J = H^{-2}(EN - 2FM + GL), \tag{3.1}$$

and the second (or total) curvature, the product of the two principal curvatures, by

$$K = H^{-2}(LN - M^2). \tag{3.2}$$

Since with the definitions (2.8), $H^2 = 1 - F^2$, we have

$$H\, dH/dt = -F\, dF/dt,$$

and, since $F = 0$ at the initial instant, $\boldsymbol{\nu}$ and $\boldsymbol{\nu}'$ being perpendicular, $dH/dt = 0$. Consequently the rate of change of first curvature is

$$\frac{dJ}{dt} = \frac{dL}{dt} + \frac{dN}{dt} - 2M\frac{dF}{dt}, \tag{3.3}$$

and, for the second curvature,

$$\frac{dK}{dt} = L\frac{dN}{dt} + N\frac{dL}{dt} - 2M\frac{dM}{dt}. \tag{3.4}$$

In the calculation of the rates of change of the curvatures of the entrainment interface, the following result will be used. If $\mathbf{x}$ and $\mathbf{x}+\mathbf{l}$ are neighbouring points on the interface, then

$$n_i(\mathbf{x}+\mathbf{l}) - n_i(\mathbf{x}) = l_k \frac{\partial n_i}{\partial x_k} = l\nu_k \frac{\partial n_i}{\partial x_k},$$

so

$$\frac{d}{dt} n_i(\mathbf{x}+\mathbf{l}) - \frac{d}{dt} n_i(\mathbf{x}) = l_k \frac{\partial}{\partial x_k}\left(\frac{dn_i}{dt}\right)$$

$$= \frac{dl}{dt}\nu_k \frac{\partial n_i}{\partial x_k} + l\frac{d}{dt}\left(\nu_k \frac{\partial n_i}{\partial x_k}\right).$$

Consequently,

$$\frac{d}{dt}\left(\nu_k \frac{\partial n_i}{\partial x_k}\right) = \nu_k \frac{\partial}{\partial x_k}\left(\frac{dn_i}{dt}\right) - \frac{1}{l}\frac{dl}{dt}\nu_k \frac{\partial n_i}{\partial x_k}. \tag{3.5}$$

Now $$L = -\nu_i \nu_j \partial n_i/\partial x_j,$$

so that $$\frac{dL}{dt} = -\nu_i \frac{d}{dt}\left(\nu_j \frac{\partial n_i}{\partial x_j}\right) - \nu_j \frac{\partial n_i}{\partial x_j}\frac{d\nu_i}{dt}.$$

With this form and (3.5), the use of (2.13), (2.10) and (2.4) leads to

$$\frac{dL}{dt} = n_l \nu_i \nu_k \frac{\partial}{\partial x_k}\left\{(\nu_i \nu_m + \nu'_i \nu'_m)\frac{\partial u_l}{\partial x_m}\right\} - 2L\nu_i \nu_j \frac{\partial u_i}{\partial x_j} + (L^2 - M^2)V.$$

Similarly

$$\frac{dN}{dt} = n_l \nu'_i \nu'_k \frac{\partial}{\partial x_k}\left\{(\nu_i \nu_m + \nu'_i \nu'_m)\frac{\partial u_l}{\partial x_m}\right\} - 2N\nu'_i \nu'_j \frac{\partial u_i}{\partial x_j} + (N^2 - M^2)V.$$

Consequently, from (3.3) and (2.17)

$$\frac{dJ}{dt} = n_l D_i D_i u_l - 2\nabla_i u_i + (L^2 + 2M^2 + N^2)V, \tag{3.6}$$

where the operators

$$\left.\begin{aligned} D_i &= (\nu_i \nu_j + \nu'_i \nu'_j)\partial/\partial x_j, \\ \nabla_i &= \nu_i\left(L\nu_k \frac{\partial}{\partial x_k} + M\nu'_k \frac{\partial}{\partial x_k}\right) + \nu'_i\left(N\nu'_k \frac{\partial}{\partial x_k} + M\nu_k \frac{\partial}{\partial x_k}\right). \end{aligned}\right\} \tag{3.7}$$

Since $\boldsymbol{\nu}$ and $\boldsymbol{\nu}'$ are initially perpendicular, $J = L + N$ at this instant, and (3.6) can be written alternatively as

$$\frac{dJ}{dt} = n_l D_i D_i u_l - 2\nabla_i u_i + (J^2 - 2K)V. \tag{3.8}$$

Also, since $$M = -\nu'_i\left(\nu_j \frac{\partial n_i}{\partial x_j}\right),$$

it can similarly be shown that

$$\frac{dM}{dt} = n_l \nu'_i \nu_k \frac{\partial}{\partial x_k} D_i u_l - M\theta,$$

where θ is the rate of surface dilatation, and

$$\frac{dK}{dt} = n_l \nabla^*_i D_i u_l - 2K\theta + JKV, \tag{3.9}$$

where the additional operator

$$\nabla^*_i = \nu_i\left(N\nu_k \frac{\partial}{\partial x_k} - M\nu'_k \frac{\partial}{\partial x_k}\right) + \nu'_i\left(L\nu'_k \frac{\partial}{\partial x_k} - M\nu_k \frac{\partial}{\partial x_k}\right). \tag{3.10}$$

These results, (3.6) and (3.9) are closely related to some given by Weatherburn (1927) in a study of the infinitesimal deformations of surfaces.

At the entrainment interface, the velocity field and its derivatives experienced by an element of the moving surface are random functions of time, so that (3.6) and (3.9) can be expressed as

$$\left.\begin{aligned} \frac{dJ}{dt} &= f_1(t) + (J^2 - 2K)V, \\ \frac{dK}{dt} &= f_2(t) - 2K\theta(t) + JKV. \end{aligned}\right\} \tag{3.11}$$

Note that the scalar functions $f_1(t)$ and $f_2(t)$ are to be evaluated at each instant by means of the definitions (3.7) and (3.10) in co-ordinates instantaneously orthogonal, but they are independent of the orientation of the co-ordinates. In the form (3.11), then, the equations can, in principle, be integrated to give J and K as functions of time for an element of the interface. Note also that even if the velocity field and its derivatives are *stationary* random functions of time, the functions f_1 and f_2 are not necessarily statistically stationary, because of the factors L, M and N in the definitions of ∇_i and ∇_i^*.

Although it does not seem possible to find general analytical solutions to (3.11), some qualitative information can be obtained readily. The first specifies dJ/dt as the sum of a randomly fluctuating quantity and a term which, from (3.6), is essentially positive and of order $J^2 V$. Consider, then, the equation

$$dJ/dt = f_1(t) + \mathscr{V} J^2,$$

where $\mathscr{V} > 0$. This can be re-written as

$$\frac{d}{dt}(J^{-1}) = -\mathscr{V} - J^{-2} f_1(t). \tag{3.12}$$

Now if, for a particular element of the interface, $J^{-1} > 0$ (as in a 'valley' between turbulent billows) its subsequent development is represented by a linear decrease superimposed on which there is a random variation whose expected amplitude diminishes as J^{-2}. Consequently, there is the expectation that *within a finite time interval*, J^{-1} will vanish and the first curvature J become infinite. The smooth valley will develop into a sharp re-entrant wedge. Only if $f_1 < -\mathscr{V} J^2$ will this progression be halted, the convective motion overcoming the tendency of the advancing front to develop sharp corners, though, if f_1 continues to vary randomly, the pause can be expected to be only temporary. On the other hand, if $J^{-1} < 0$ as on the surface of a turbulent billow, it will tend to become more negative. The first curvature, if negative, will decrease in absolute value but tend to remain negative. However, as J^{-1} decreases, the expected amplitude of the fluctuating term $J^{-2} f(t)$ increases, and the likelihood of J changing sign increases; when this happens, a further re-entrant wedge can develop. The characteristic geometry of the entrainment interface is one in which regions of predominantly negative curvature (billows) are separated by lines of very large positive curvature (wedges). As the surface evolves, new wedges can be expected to appear on the surface of the billows as local regions of positive curvature develop as a result of the turbulent motion.

It has already been pointed out that the billows represent, on average, *sources* of surface area and the wedges *sinks*. On a billow, with $J < 0$, one would expect intuitively that the surface divergence θ would generally be positive, but, even if the mean value of θ over the billow vanishes, (2.16) shows that

$$\overline{\frac{1}{A}\frac{dA}{dt}} = -V\bar{J} > 0.$$

The process of entrainment simply expands both the volume and the surface area of a billow. On the other hand, when $J > 0$, the surface area tends to contract; in the limiting geometry of a wedge, surface is destroyed as the two faces of the wedge advance on each other with speed V. Reference to figure 2 shows that

$$\frac{\delta r}{V\delta t} = -\frac{1}{\sin 2\alpha},$$

where α is the semi-angle of the wedge. Consequently, if α is a function of distance s along the apex of the wedge, the rate of loss of area

$$\left(\frac{dA}{dt}\right)_w = -2V\int\frac{ds}{\sin 2\alpha}. \tag{3.13}$$

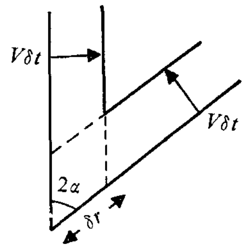

FIGURE 1. The advance of a re-entrant wedge.

If $\mathscr{L}$ represents the total length of the wedge lines in a large area A of the entrainment interface, then

$$\left(\frac{dA}{dt}\right)_w = -2V\mathscr{L}\,\overline{\operatorname{cosec} 2\alpha},$$

and

$$\overline{\left(\frac{1}{A}\frac{dA}{dt}\right)_w} = -2V\Gamma\,\overline{\operatorname{cosec} 2\alpha},$$

where Γ is the average length of wedge lines per unit area of the interface. If the geometry of the entrainment interface is statistically stationary (though in real turbulent flows there seems no reason to suppose this to be generally true), this mean rate of destruction of area per unit surface area by the wedges would be balanced by the mean rate of generation in the bulges, given by (2.16) averaged over the surface area of these regions.

4. An Eulerian description

The turbulent interface can be represented by the surface $z = \zeta(x, y, t)$, every element of which is advancing normal to itself with speed V while the whole is being convected and convoluted by the larger-scale components of the turbulent flow. For the sake of definiteness, the mean position of the entrainment interface can be taken as parallel to the x, y plane. The surface $z = \zeta$ is likely to be continuous, but it may contain discontinuities in gradient along the lines in the surface, and it may be multiply valued over domains in (x, y, t) where a billow has moved above a region in irrotational motion. The surface is convoluted by the large-scale motion and the rate of change of ζ is given by

$$\frac{d\zeta}{dt} = w_\zeta + V(\mathbf{n}\,.\,\boldsymbol{\mu}), \tag{4.1}$$

where w_ζ is the component of the fluid velocity in the z direction at the instantaneous position of the interface, $\mathbf{n}$ is the unit normal to the surface and $\boldsymbol{\mu}$ a unit vector in the z direction. This equation can be written equivalently as

$$\dot{\zeta}+\mathbf{u}_\zeta.\nabla\zeta = w_\zeta + V[1+(\nabla\zeta)^2]^{\frac{1}{2}}, \tag{4.2}$$

where $\mathbf{u}_\zeta$ represents the vector component of the fluid velocity in the x, y plane at the position (x, y, ζ). The positive root is to be taken when $\mathbf{n}.\boldsymbol{\mu} > 0$, and the negative root when $\mathbf{n}.\boldsymbol{\mu} < 0$, i.e., beneath the 'overhang' when ζ is multiply valued.

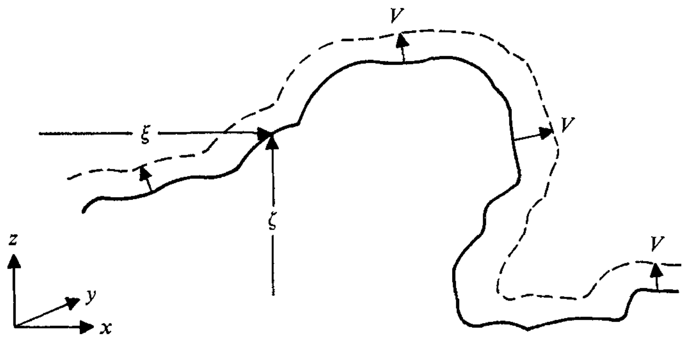

FIGURE 2. Specification of the entrainment interface.

The general problem might be posed as follows. Given an initial configuration $\zeta(x, y, t_0)$ of the entrainment interface and a random large-scale velocity field $\mathbf{q} = (\mathbf{u}, w) = \mathbf{q}(x, y, z, t)$, irrotational on one side of the interface and vortical on the other, find from (4.2) the subsequent development of ζ in space and time. The analytical difficulty arises from the fact that the velocity field of interest in (4.2) is specified at the position of the interface which is unknown *a priori*, and this obstructs the finding of explicit solutions except in some very simple (though revealing) cases. Clearly, numerical experiments could be conducted using a digital computer with step-by-step integrations in time, but this is not within the scope of the present work.

In addition to the rate at which the interface moves in the z direction, we are also interested in the rate at which it moves in the x direction, a quantity called the 'convection velocity' u_c by Kovasznay (1970). The surface can be specified equivalently by the multiply valued function $x = \xi(y, z, t)$ so that

$$\frac{d\xi}{dt} = u_\xi + V[1+(\nabla\xi)^2]^{\frac{1}{2}}, \tag{4.3}$$

where $\nabla\xi = (\partial\xi/\partial y,\ \partial\xi/\partial z)$. An equivalent sign convention holds for the square root; if $\boldsymbol{\lambda}$ is a unit vector in the x direction, the root is taken positive when $\boldsymbol{\lambda}.\mathbf{n} > 0$ and negative when $\boldsymbol{\lambda}.\mathbf{n} < 0$. Thus,

$$u_c = \dot{\xi} = u_\xi - v_\xi\frac{\partial\xi}{\partial y} - w_\xi\frac{\partial\xi}{\partial z} + V[1+(\nabla\xi)^2]^{\frac{1}{2}}, \tag{4.4}$$

where u_ξ, v_ξ and w_ξ are the velocity components in the x, y and z directions at the instantaneous position of the interface.

Some simple geometry enables us to express u_c in terms of $\zeta(x, y, t)$. Holding y fixed for small displacements $dz = d\zeta$ and $dx = d\xi$,

$$d\xi/dz = (\partial\zeta/\partial x)^{-1}.$$

Also, when z is fixed, a small variation along $\zeta(x, y) =$ constant gives

$$\frac{\partial\zeta}{\partial x}\delta\xi + \frac{\partial\zeta}{\partial y}\delta y = 0,$$

so that

$$\frac{\partial\xi}{\partial y} = -\frac{\partial\zeta}{\partial y}\bigg/\frac{\partial\zeta}{\partial x}.$$

With these expressions and (4.4), we have

$$u_c = u_\zeta + \left(\frac{\partial\zeta}{\partial x}\right)^{-1}\left\{v_\zeta\frac{\partial\zeta}{\partial y} - w_\zeta - V[1+(\nabla\zeta)^2]^{\frac{1}{2}}\right\}, \tag{4.5}$$

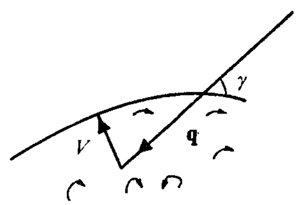

FIGURE 3. The passage of a streamline through the interface.

where the positive root is to be taken. A check on these expressions can be made by substitution of the left-hand side of (4.2) into (4.5), which leads to the identity

$$u_c = -\frac{\partial\zeta}{\partial t}\bigg/\frac{\partial\zeta}{\partial x}. \tag{4.6}$$

Similar expressions can, of course, be found for the motion of the interface in the y direction.

An alternative way of describing the entrainment interface is useful in examples (admittedly artificial ones) in which there exists a frame of reference in which both the flow field and the entrainment interface are steady in time. In such a frame, the streamlines and particle paths coincide; the normal component of the fluid velocity across the interface balances the entrainment speed V. If q_ζ is the speed of the fluid crossing the interface and γ the angle between the streamline and the interfacial surface then, as shown in figure 3,

$$\sin\gamma = V/q_\zeta. \tag{4.7}$$

Thus, $q_\zeta \sin\gamma = V$, a constant along the interface.

5. Some simple examples

Although the general problem of finding $\zeta(x, y, t)$ from (4.2) is peculiarly intractable, some valuable insight into the general characteristics of the entrainment interface can be gained by examining the properties of some simple cases.

We shall be interested not so much in the detailed form of the solutions but in such quantities as the overall speed of advance, the surface curvature and the conditions under which there appear discontinuities in surface slope. The analytical difficulties are reduced considerably when the large-scale velocity field $\mathbf{q}$ at the interface is independent of the location ζ of the interface.

5.1. *The case* $u = 0$, $w = Wf(x)$.

In this case, the large-scale velocity field is two-dimensional, normal to the interface and steady in time. The *maximum* value of the velocity is taken as W, so that $f(x) \leqslant 1$. Solutions of the form $\zeta = \zeta(x, t)$ exist, where from (4.2)

$$\zeta_t = Wf(x) + V[1 + \zeta_x^2]^{\frac{1}{2}}. \tag{5.1}$$

Note that if $f(x)$ attains its maximum value of unity at isolated points, then in the vicinity of such a point, $x = x_0$, say, $f(x)$ is an even function of $(x - x_0)$. Consequently, ζ is also an even function of $(x - x_0)$: the maximum in the interfacial configuration coincides with the maximum velocity in the z direction. At this point, $\zeta_x = 0$ and the speed of advance

$$\zeta_t(x_0) = W + V,$$

the sum of the convection velocity and the entrainment speed V.

If the turbulent interface is initially plane and the function f is, say, periodic in x, the interface advances steadily with speed $W + V$ at the points $x = x_0$ of maximum w and at other points recedes relative to this. We can therefore let

$$\zeta = (W + V)t + \Phi(x, t), \tag{5.2}$$

and, from (5.1),

$$\Phi_t = -W(1 - f) + V[(1 + \Phi_x^2)^{\frac{1}{2}} - 1]. \tag{5.3}$$

Initially, $\Phi_t = -W(1 - f) < 0$ at all points save $x = x_0$, and the surface develops slope. Consequently, the magnitude of the last term in (5.3) increases steadily and Φ_t decreases. Ultimately $\Phi_t = 0$ when

$$V[(1 + \Phi_x^2)^{\frac{1}{2}} - 1] = W(1 - f), \tag{5.4}$$

and a steady configuration of the advancing interface is attained. With a prescribed distribution $f(x)$, the asymptotic shape of the surface is given by the solution of the differential equation

$$\Phi_x^2 = \frac{W^2}{V^2}(1 - f)^2 + \frac{2W}{V}(1 - f) \tag{5.5}$$

subject to the conditions $\Phi(x_0) = 0$ and $\Phi_{xx}(0) < 0$ since ζ has a local maximum at this point.

Alternatively, the angle of slope of the interfacial surface can be determined immediately. If $\Phi_x = \tan\beta(x)$ then from (5.4),

$$V(\sec\beta - 1) = W(1 - f),$$

or

$$\beta(x) = \sec^{-1}\left\{1 + \frac{W}{V}[1 - f(x)]\right\}, \tag{5.6}$$

the negative value being taken when $x > 0$ and vice versa.

For example, if $f(x) = \cos kx$ then, from (5.5),

$$\Phi_x^2 = 4\Gamma^2 \sin^4 \tfrac{1}{2}kx + 4\Gamma \sin^2 \tfrac{1}{2}kx,$$

where $\Gamma = W/V$, and

$$\Phi(x) = -(\Gamma k^{-1}) \int_0^{\frac{1}{2}kx} \sin u (\sin^2 u + \Gamma^{-1})^{\frac{1}{2}} du, \tag{5.7}$$

when $-\pi < \frac{1}{2}kx < \pi$. The shapes of the entrainment interface are illustrated in figure 4 for various values of Γ, and, as expected, are characterized by re-entrant wedges whose semi-angle can be found either from (5.5) or (4.7):

$$\alpha = \sin^{-1}\left(\frac{1}{2\Gamma+1}\right) = \sin^{-1}\left(\frac{V}{2W+V}\right).$$

The greater the value of Γ, the deeper the wedges and the sharper the re-entrant angles. The curvature of the bulges is negative everywhere, except for the discontinuities.

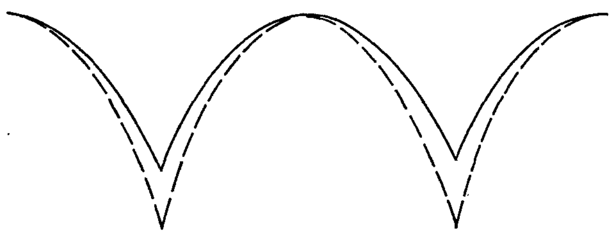

FIGURE 4. The configuration of the interface for a non-propagating sinusoidal disturbance. The unbroken line indicates the shape when $\Gamma = W/V = 1$; the broken line when $\Gamma = 2$. Vertical distortion 0·5.

Consider now the behaviour of a small perturbation to a steady solution of (5.3). Let $\Phi = \Phi_0(x) + \phi(x,t)$, where $\Phi_0(x)$ satisfies (5.4). By substitution into (5.3) it can be shown simply that

$$\frac{\partial \phi}{\partial t} - V \sin \beta(x) \frac{\partial \phi}{\partial x} = 0, \tag{5.8}$$

where β is the undisturbed angle of surface slope, so that

$$\sin \beta(x) = \frac{\partial \Phi_0}{\partial x}\left\{1 + \left(\frac{\partial \Phi_0}{\partial x}\right)^2\right\}^{\frac{1}{2}}$$
$$= \frac{\partial \zeta}{\partial s}$$

in the undisturbed state, ds being an element of length along the surface. Equation (5.8) indicates clearly that small disturbances to the surface propagate along it, the component of the propagation velocity in the x direction being $-V \sin \beta(x)$. Disturbances in the vicinity of a maximum in ζ will move away from the maximum, towards either the next smooth minimum in ζ or the next re-entrant corner. If a corner is encountered, such as those illustrated in figure 4 where $\sin \beta$ changes sign discontinuously, the disturbance reaches the corner with finite velocity and is consumed there. The surface recovers its original configuration,† and, in this sense, solutions with alternating smooth maxima and sharp minima can be said to be stable.

Solutions to (5.5) do, of course, exist with smooth minima in Φ, e.g. (5.7) defined over all x. Such solutions are excluded by the initial condition that $\Phi(x,0) = 0$, and moreover they are unstable. Small disturbances *approach* a smooth minimum with a velocity that decreases to zero at the minimum. They therefore accumulate in this region with ever-increasing slope, and the surface never returns to its original configuration.

5.2. *The case* $u = -U$ *(a constant)*, $w = Wf(x)$.

This represents a normal velocity field moving with speed U along the surface. As before, we define W to be the maximum value of w, so that $f \leqslant 1$. Let

$$\zeta = \mathscr{W}t + \Phi(x,t),$$

where $\mathscr{W}$ is the speed of advance of the front as a whole. From (4.2),

$$\mathscr{W} + \Phi_t - U\Phi_x = Wf(x) + V[1+\Phi_x^2]^{\frac{1}{2}}. \tag{5.9}$$

The speed $\mathscr{W}$ cannot now be specified immediately, as in the previous example, since the maxima in ζ and w do not in general coincide. Indeed, the determination of $\mathscr{W}$ is the principal aim of the present analysis. If ζ has a maximum at the point x_0, so that $\Phi_x(x_0) = 0$, then, in a steady configuration, (5.9) indicates that

$$\mathscr{W} = Wf(x_0) + V < W + V.$$

Also, if f (and Φ_x) are either integrable over the whole x domain or periodic with zero mean, then the average of (5.9) gives

$$\mathscr{W} = \overline{V[1+\Phi_x^2]^{\frac{1}{2}}} \geqslant V,$$

equality holding only when f is integrable. Consequently,

$$W + V > \mathscr{W} \geqslant V, \tag{5.10}$$

but within this range $\mathscr{W}$ remains to be determined by the solution.

The behaviour of small perturbations to a steady solution of (5.9) can be found as before by letting $\Phi = \Phi_0(x) + \phi(x,t)$, where $\Phi_0(x)$ satisfies (5.9). The disturbance ϕ satisfies the equation,

$$\frac{\partial \phi}{\partial t} - (U + V\sin\beta(x))\frac{\partial \phi}{\partial x} = 0, \tag{5.11}$$

and so propagates in the x direction with velocity $-(U+V\sin\beta)$. A local disturbance in some neighbourhood will therefore move along the surface, leaving its configuration ultimately unchanged if this quantity does not vanish at any point, and so remains of the same sign. Taking $U > 0$, a *sufficient* condition for the existence of a steady solution with continuous derivatives is then that

$$U > V. \tag{5.12}$$

† Note that, if ϕ is differentiable and non-zero at the maximum of ζ, the whole front will advance an amount equal to the value of ϕ at that point, but the shape will remain unchanged.

The most interesting situations, however, are those in which $U + V\sin\beta$ does change sign. If this quantity becomes negative at one point, and if the slope is continuous, it must become positive again at at least one other point, since the average value of β along the surface vanishes. A point where $U + V\sin\beta$ becomes negative as x increases is such that disturbances propagate *away from* this point; it will be called a *leading point* on the surface. A point where $U + V\sin\beta$ becomes positive is a point of accumulation of small disturbances; such a point is unstable in the sense described above, and the slope tends to become discontinuous. We would therefore anticipate that the configuration of the entrainment interface in these cases will involve one or more leading points, separated by discontinuities, where $U + V\sin\beta(x)$ abruptly changes sign from negative to positive. A *necessary* condition for this type of configuration is clearly that $V \geqslant U$; it will later be shown to be sufficient.

This behaviour can be shown more explicitly by rewriting (5.9) in terms of the angle of slope $\beta(x)$. In the steady state, since $\Phi_x = \tan\beta(x)$, we have

$$\mathscr{W} - U\tan\beta = Wf + V\sec\beta, \tag{5.13}$$

and on differentiation with respect to x, there results

$$\frac{d\beta}{dx} = -\frac{Wf'(x)\cos^2\beta}{U + V\sin\beta}. \tag{5.14}$$

If $U + V\sin\beta(x) > 0$ everywhere, the coefficient of $f'(x)$ is everywhere finite and negative. The points of greatest negative and positive *slope* now coincide with the maxima and minima in the distribution of the normal velocity field. At a leading point, however, the denominator vanishes, and, since the slope must be continuous at this point, $f' = 0$ also. Consequently, at the leading point,

$$\frac{d\beta}{dx} = \frac{-Wf''(x)\cos^2\beta}{V\cos\beta(d\beta/dx)},$$

so that

$$\left(\frac{d\beta}{dx}\right)^2 = -\frac{W}{V}f''(x)\cos\beta. \tag{5.15}$$

Since $\cos\beta > 0$, $f(x)$ must have its *maximum* value (of unity) at a leading point. Note, however, that a leading point does not generally coincide with a maximum of ζ (the crest of a billow).

The existence of a leading point enables us to find the entrainment speed $\mathscr{W}$ in a simple manner. From (5.13),

$$\mathscr{W} - Wf(x) = \frac{U\sin\beta(x) + V}{\cos\beta(x)},$$

and, if the point $x = x_1$ is a leading point, $\sin\beta = -U/V$, so that

$$\mathscr{W} - W = \frac{V^2 - U^2}{V\cos\beta},$$

since $f(x_1) = 1$. Thus,

$$\tan\beta(x_1) = \frac{\sin\beta(x_1)}{\cos\beta(x_1)} = \frac{U(\mathscr{W} - W)}{U^2 - V^2}. \tag{5.16}$$

Also, from (5.9), with $\Phi_t = 0$,

$$\mathscr{W} - Wf(x) - U\Phi_x = V[1+\Phi_x^2]^{\frac{1}{2}}, \tag{5.17}$$

so that

$$(U^2 - V^2)\Phi_x^2 - 2U(\mathscr{W} - Wf)\Phi_x + (\mathscr{W} - Wf)^2 - V^2 = 0, \tag{5.18}$$

and, if $U^2 - V^2 \neq 0$,

$$(U^2 - V^2)\Phi_x = U(\mathscr{W} - Wf) \pm V\{U^2 - V^2 + (\mathscr{W} - Wf)^2\}^{\frac{1}{2}}. \tag{5.19}$$

Comparison of this expression with (5.16) indicates that, at a leading point, where $f = 1$, the radical vanishes. Thus,

$$U^2 - V^2 + (\mathscr{W} - W)^2 = 0$$

and

$$\mathscr{W} = W + (V^2 - U^2)^{\frac{1}{2}}. \tag{5.20}$$

The surface slope at the leading point is found immediately from (5.16) and (5.20) to be

$$\tan\beta(x_1) = \frac{-U}{(V^2-U^2)^{\frac{1}{2}}}, \tag{5.21}$$

which is, rather surprisingly, independent of $Wf(x)$. Note that, if the leading point is to exist it is *necessary* that $V \geqslant U$. As $U \to V^-$, the negative slope at the leading point becomes indefinitely large, the velocity of advance V of the vertical element of front just balancing the convective velocity $-U$. Note also that, when $V \geqslant U$, (5.19) and (5.20) show that $f = 1$ necessarily corresponds to a leading point. Therefore, the condition $V \geqslant U$ is necessary and sufficient for the existence of leading points (and discontinuities in slope); the condition $U > V$ is necessary and sufficient for regular solutions.

At a leading point, (5.15) shows that the curvature of the surface is finite and non-zero. Consequently, Φ_x is locally an odd function about its value (5.21) at the point, so that the sign of the radical in the solution (5.19) changes as we pass through this point, and the value of the radical vanishes. From (5.17), when $\Phi_x = 0$, $\mathscr{W} - Wf = V$, so that the negative sign in the solution is required to the left of the leading point and the positive sign between the leading point and the next discontinuity in slope. When $U > V$, there are no leading points, the negative sign being required throughout.

For any particular distribution f, the surface configuration can be found when $V > U$ by integrating in both directions away from the leading point, using the initial slope (5.21). If f is periodic, the solutions will ultimately intersect the curve associated with the neighbouring cycle and discontinuities in slope appear as in the previous example with $U = 0$. Some representative shapes are shown in figure 5.

Equation (5.20), specifying the overall speed of advance when $U \leqslant V$, is the primary result of this section. When $U > V$, there are no leading points, and this method cannot be used. The solutions to (5.19) are, however, continuous, and they can be integrated over the entire range:

$$(U^2 - V^2)\Phi(x) = U\int^x \{\mathscr{W} - Wf\}\,dx - V\int^x \{U^2 - V^2 + (\mathscr{W} - Wf)^2\}^{\frac{1}{2}}\,dx, \tag{5.22}$$

to an arbitrary additive constant. In this expression, $\mathscr{W}$ is still to be determined. If f is a spatially periodic function, such that $f(x) = f(x+2\pi)$, and with zero mean (so that there is no net outflow in the z direction), then Φ is similarly periodic, and

$$2\pi U\mathscr{W} = V\int_0^{2\pi}\{U^2 - V^2 + (\mathscr{W} - Wf)^2\}^{\frac{1}{2}}\,dx, \tag{5.23}$$

an implicit relation for $\mathscr{W}$ as a function of U, V, W and the distribution $f(x)$. The finding of an explicit expression for $\mathscr{W}$ is in general rather difficult, but, for rapidly travelling disturbances such that $U \gg V, W$, it can be shown that

$$\mathscr{W} \simeq V\left\{1 + \frac{W^2\overline{f^2}}{2U^2}\right\},$$

where $\overline{f^2}$ is the average value of f^2 over a cycle. The velocity of advance is only slightly larger than the entrainment velocity, and much less than W.

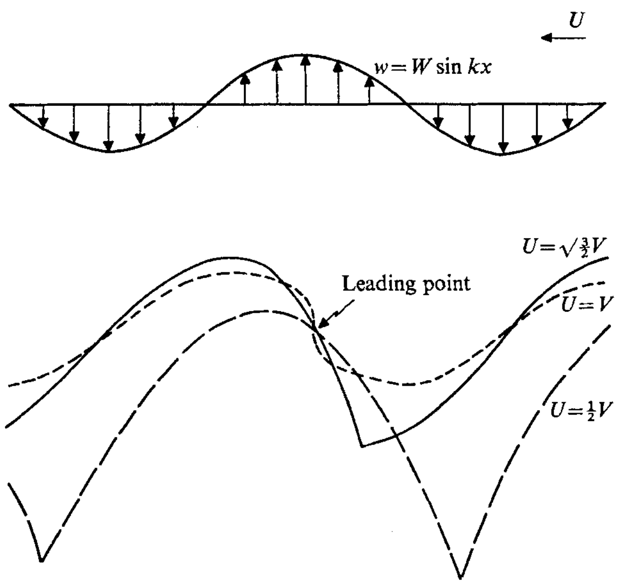

FIGURE 5. Interfacial configurations for a disturbance moving with velocity relative to the fluid at the interface, $V/W = 1$. The leading point occurs at the point of maximum outflow, and does not coincide with the crest of the billow, unless $U = 0$ (figure 4). No vertical distortion.

Another simple, though very artificial, example is provided when $f(x)$ is a square wave:

$$f(x) = \begin{cases} 1 & (0 < x < \pi) \\ -1 & (\pi < x < 2\pi). \end{cases}$$

It is then found simply from (5.23) that

$$\mathscr{W} = \frac{V}{U}(U^2 + W^2)^{\frac{1}{2}}. \tag{5.24}$$

The distribution is not, of course, differentiable, so that the previous discussion of leading points does not apply directly, but nevertheless it is interesting to note that, as $U \to V$, $\mathscr{W} \to (U^2+W^2)^{\frac{1}{2}} \simeq W$ when $V \ll W$. The speed of advance $\mathscr{W}$ of the front is comparable with W only when $U \sim V$.

A further example of interest is when $f(x)$ is integrable so that necessarily $f(x) \to 0$ as $x \to \pm\infty$. The normal velocity disturbance is limited to a local region near the origin, while far away the entrainment interface will be supposed flat and therefore advancing with speed V in the z direction. The overall entrainment velocity is then simply V, but the interface experiences a net advance as a result of the movement of the disturbance field along it. From (5.22), the net advance when $U > V$ is

$$\begin{aligned}\delta &= \Phi(-\infty) - \Phi(\infty) \\ &= (U^2 - V^2)^{-1}\left\{\int_{-\infty}^{\infty} [V\{U^2 - V^2 + (V - Wf)^2\}^{\frac{1}{2}} - U(V - Wf)]\,dx\right\}. \end{aligned} \quad (5.25)$$

5.3. *The case* $u = -U, w = Wf(x,y)$

This situation represents a normal velocity field varying in both x and y, and moving along the surface with speed U relative to the fluid outside. The configuration of the advancing interface is specified by the equation

$$\mathscr{W} + \Phi_t - U\Phi_x = Wf(x,y) + V[1 + \Phi_x^2 + \Phi_y^2]^{\frac{1}{2}}. \quad (5.26)$$

Although it is not easy to find analytical solutions to this nonlinear partial differential equation, the concept of leading points when $U \leqslant V$ will be found to generalize immediately and enables us to find the overall speed of advance $\mathscr{W}$.

As before suppose that $\Psi(x,y)$ represents a steady-state solution to (5.26), and consider the behaviour of a small perturbation. If $\Phi = \Psi(x,y) + \phi(x,y,t)$, then, for infinitesimal disturbances, it is found that

$$\frac{\partial \phi}{\partial t} - \left\{U + \frac{V\Psi_x}{[1 + \Psi_x^2 + \Psi_y^2]^{\frac{1}{2}}}\right\}\frac{\partial \phi}{\partial x} - \frac{V\Psi_y}{[1 + \Psi_x^2 + \Psi_y^2]^{\frac{1}{2}}}\frac{\partial \phi}{\partial y} = 0. \quad (5.27)$$

The propagation velocity of small disturbances vanishes when

$$\left.\begin{aligned}\Psi_y &= 0, \\ U + V\Psi_x[1 + \Psi_x^2]^{-\frac{1}{2}} &= U + V\sin\beta = 0,\end{aligned}\right\} \quad (5.28)$$

where β is the angle of slope of the section in the x,z plane. The first of these equations indicates that, locally, the surface is normal to the x,z plane; if Ψ_{xx}, $\Psi_{yy} < 0$, small disturbances propagate in all directions *away from* such points, and they are therefore stable leading points in the sense described before. The previous analysis of this region applied unchanged, and the speed of advance is given as in (5.20) by $\mathscr{W} = W + (V^2 - U^2)^{\frac{1}{2}}$. The projection of the entrainment interface on the x,y plane defines a generally irregular mosaic of closed domains, each containing a leading point and bounded by the lines that are the projections of re-entrant wedges in the surface.

When $U > V$, the leading points disappear, and, as before, the entrainment rate $\mathscr{W}$ can be expected to decrease rapidly below the value W when $U = V$ until asymptotically, when $U/V \gg 1$, $W \to V$, the surface convolutions being levelled out.

6. General conclusions

The models described in the previous sections are of course rather artificial, but perhaps not to the extent that we cannot draw valuable conclusions from them. The velocity field in the z direction at the interface was supposed independent of time and (in § 5.3) an arbitrary function of x and y. In a real turbulent flow, the structure of the large-scale eddies is, of course, not independent of time, though (as the experiments of Favre, Gaviglio & Dumas 1957, 1958, 1967 indicate) the time scale of their evolution is surprisingly large. Also, in these simple solutions, the u and v fluctuations in the velocity field at the interface are neglected, together with variation of the mean velocity with z; these can be expected to distort the interfacial shapes somewhat, without, however, contributing greatly to the overall process of entrainment. Consequently, it seems not at all unreasonable to compare the predictions of the analysis with the experimental results of Kovasznay *et al.* (1970) in at least a semi-quantitative way.

One of their noteworthy findings was that, in the outer part of a boundary layer, the turbulent billows move rather more slowly than the outside stream. Making observations at a fixed point, they found that the convection velocity (in the x direction) of the interface at the 'front' of a billow as it travelled past was rather greater than that at the 'back', because of the expansion and outwards movement of the billow. Both, however, were less than U_∞; the average of these two convection velocities, which we can take as the propagation speed of the billow was about $0{\cdot}94U_\infty$, when the intermittency factor $\gamma = 0{\cdot}5$ decreasing to $0{\cdot}92U_\infty$ at points deeper in the boundary layer, where $\gamma \sim 0{\cdot}9$. The propagation speed of the large billows relative to the outside fluid was then about $0{\cdot}06U_\infty$, or, in terms of the friction velocity u_*, about $1{\cdot}3\,u_*$. Now, according to the theory, only those disturbances propagating at speeds less than V relative to the outside fluid are capable of producing large convolutions of the surface. It follows immediately that for this turbulent boundary layer, V is greater than $1{\cdot}3\,u_*$.

A closer estimate can be obtained from the measurements of the average normal (w) velocity component at different points of the interface (Kovasznay *et al.* 1970, figure 9). Over the 'fronts' of the billows, the average velocity was found to be negative for all positions of the interface, indicating a mean motion towards the wall. On the 'backs' the mean velocity was positive or outwards when the intermittency factor γ was less than $0{\cdot}6$ (i.e. towards the top of the billows), but negative when $\gamma > 0{\cdot}6$, in the valleys. It is interesting to consider the results of figure 5 in the light of these findings. In this figure, the left-hand portion of the billow corresponds to 'fronts' and the right-hand portion to the 'backs' of Kovasznay *et al.*; the distribution of normal velocity relative to the billow shape is very similar to that calculated when the convection velocity $U = \frac{1}{2}V$. In this case, the leading point, where the outwards velocity is greatest, is on the rear side of the billow and the region of outwards velocity extends from just ahead of the crest to a point on the back of the billow well in the valley, where in fact $\gamma \sim 0{\cdot}75$. Over virtually all of the front face, as well as on the back face, when $\gamma > 0{\cdot}75$, the w velocity component is negative. This behaviour is very similar to that observed in the experiments and contrasts with the situation when

$U = V$, when the normal velocity is outwards on the whole of the back face and inwards on the front face, and when $U = \sqrt{\frac{3}{2}}V$ and the outwards velocity region includes the valley and some of the front face. When $U = 0$ (as in figure 4), the outwards velocity and surface shape are symmetrically disposed, in clear conflict with the experimental results. This comparison suggests quite strongly that, in the turbulent boundary layer, the convection velocity ($1{\cdot}3u_*$) is close to $\frac{1}{2}V$, or that the entrainment speed is approximately $2{\cdot}6u_*$.

This figure is surprisingly large, especially when expressed in terms of the Kolmogorov velocity u_ν. Townsend's (1956) results indicate that near the outer edge of a boundary layer, the energy dissipation rate $\epsilon_0 \sim u_*^3/\delta$, so that

$$\frac{u_\nu}{u_*} = \frac{(\epsilon_0 \nu)^{\frac{1}{4}}}{u_*} \sim \left(\frac{U_\infty}{u_*}\right)^{\frac{1}{4}} \left(\frac{\nu}{U_\infty \delta}\right)^{\frac{1}{4}}.$$

In the experiments of Kovasznay *et al.*, $u_*/U_\infty = 0{\cdot}045$ and $U_\infty \delta/\nu = 27500$, so that $u_\nu/u_* = 0{\cdot}17$, approximately. Consequently, in these experiments $V \simeq 15\,u_\nu$. Corrsin & Kistler's argument (described in §1) would lead us to expect that $V \propto u_\nu$, but the numerical coefficient 15 is considerably larger than the 'order unity' value usually expected (and found!) in reasoning based on the theory of local similarity.

However, it must be remembered that, although the description of this theory has been couched in terms of disturbances with length scales large compared with the Kolmogorov scale in the turbulence, the kinematics involved are the same for the whole range of disturbances from the largest scale to those only slightly larger than the thickness of the interface. If viewed on a small scale, the entrainment interface will have micro-convolutions as a result of the continuous distribution of scales deforming it, and whether or not these are detected separately or seen only as a 'blurring' of the interface depends on the response of the detecting system. Indeed, the 'thickness' of the entrainment interface itself depends on this response, provided that it is larger than the Kolmogorov microscale, and the entrainment speed V, augmented by micro-convolutions in the same way that the overall speed of advance is increased by large-scale convolutions, is correspondingly numerically larger than u_ν. The statistical geometry of the interface is, of course, extremely nonlinear and the contributions from different ranges of eddies cannot be superimposed. The speed of advance of a given element of the interface on a billow, when viewed on a large scale, depends essentially on the *maximum* normal velocity found in the element, or in a neighbouring one from which disturbances can propagate.

Turbulent entrainment is an extremely important process in both meteorology and oceanography. In these contexts, the entrainment interface is frequently accompanied by a density jump whose influence on the large-scale motion will modify the entrainment process. For example, at the top of the mixed layer in the atmosphere, or at the bottom of it in the ocean, the density jump is stable, and the larger-scale disturbances will tend to propagate as internal gravity waves. While the local entrainment speed V may be only slightly reduced, the speed of propagation of the larger disturbances may well considerably exceed V, so that the billowing characteristic of wind-tunnel turbulence disappears, and the overall speed of advance of the turbulent front is reduced substantially. The quantitative discussion of these matters is, however, beyond the scope of the present paper.

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